AAECC **10**, 237–250 (2000)

AAECC
Applicable Algebra in
Engineering, Communication
and Computing

Computation of the Dimensions of Symmetry Classes of Tensors Associated with the Finite two Dimensional Projective Special Linear Group

M. R. Darafsheh, M. R. Pournaki

Department of Mathematics and Computer Science, University of Tehran
& Institute for Studies in Theoretical Physics and Mathematics, Tehran, Iran
(e-mail: darafshe, pournaki@vax.ipm.ac.ir)

Received: October 14, 1998; revised version: November 30, 1999

Abstract. The dimensions of the symmetry classes of tensors associated with the projective special linear group of degree 2 over a field with q elements, $PSL_2(q)$, are found. Of course we will assume $PSL_2(q)$ as a subgroup of the symmetric group S_{q+1} because this group has a faithful action on the points of the underlying projective space. We also discuss the non-triviality of the symmetry classes of tensors associated with each irreducible character of $PSL_2(q)$.

Keywords: Symmetry classes of tensors, Actions of symmetric groups, Irreducible characters, Projective special linear group.

1 Introduction

Let V be an s-dimensional vector space over the complex field $\mathbb{C}$. Let $\otimes^t V$ be the t-th tensor power of V and write $v_1 \otimes \cdots \otimes v_t$ for the tensor product of the indicated vectors. To each permutation σ in the symmetric group S_t there corresponds a unique linear operator $P(\sigma)\colon \otimes^t V \to \otimes^t V$ determined by $P(\sigma)(v_1 \otimes \cdots \otimes v_t) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(t)}$. Let G be a subgroup of S_t and let $I(G)$ be the set of all the irreducible complex characters of G. It follows from the orthogonality relations for characters that

$$\left\{ T(G,\chi) : \otimes^t V \to \otimes^t V \mid T(G,\chi) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) P(\sigma);\ \chi \in I(G) \right\}$$

is a set of annihilating idempotents which sum to the identity. The image of $\otimes^t V$ under $T(G,\chi)$ is called the *symmetry class of tensors* associated with G and χ and it is denoted by $V_\chi^t(G)$. It is well-known that

$$\dim V_\chi^t(G) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) s^{c(\sigma)} \tag{1}$$

where $c(\sigma)$ is the number of cycles, including cycles of length one, in the disjoint cycle decomposition of σ (see [7]).

Let Γ_s^t be the set of all sequences $\alpha = (\alpha_1, \ldots, \alpha_t)$ with $1 \leq \alpha_i \leq s$, so α is a mapping from a set of t elements into a set of s elements. Then the group G acts on Γ_s^t by $\sigma \cdot \alpha := (\alpha_{\sigma^{-1}(1)}, \ldots, \alpha_{\sigma^{-1}(t)})$ where $\sigma \in G$ is a permutation on t letters and $\alpha \in \Gamma_s^t$ is a mapping from a set of t elements into a set of s elements. Therefore the action may be written as $\sigma \cdot \alpha = \alpha\sigma^{-1}$ which is a composition of two functions. Let $O(\alpha) = \{\sigma \cdot \alpha | \sigma \in G\}$ be the *orbit* with representative α, also let G_α be the *stabilizer* of α, i.e., $G_\alpha = \{\sigma \in G \mid \sigma \cdot \alpha = \alpha\}$. Let Δ be a system of distinct representatives of the orbits of G acting on Γ_s^t and define

$$\overline{\Delta} = \left\{ \alpha \in \Delta \middle| \sum_{\sigma \in G_\alpha} \chi(\sigma) \neq 0 \right\} .$$

Let $\{e_1, \ldots, e_s\}$ be a basis of V. Denote by e_α^* the tensor $T(G, \chi)(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_t})$. For $\gamma \in \overline{\Delta}$, $V_\gamma^* = \langle e_{\sigma\cdot\gamma}^* \mid \sigma \in G \rangle$ is called the *orbital subspace* of $V_\chi^t(G)$. It follows that

$$V_\chi^t(G) = \bigoplus_{\gamma \in \overline{\Delta}} V_\gamma^* \tag{2}$$

is a direct sum. In [4] Freese proved that

$$\dim V_\gamma^* = \frac{\chi(1)}{|G_\gamma|} \sum_{\sigma \in G_\gamma} \chi(\sigma) \ , \tag{3}$$

but the above formula can be written as

$$\dim V_\gamma^* = \chi(1) \left(\chi \downarrow_{G_\gamma}, 1_{G_\gamma} \right)_{G_\gamma} \ . \tag{4}$$

If there exists $\gamma \in \Gamma_s^t$ for which $\left(\chi \downarrow_{G_\gamma}, 1_{G_\gamma} \right)_{G_\gamma} \neq 0$, then we have $\dim V_\gamma^* > 0$. Therefore by (2) we have $V_\chi^t(G) \neq 0$. So to show the non-triviality of the vector space $V_\chi^t(G)$, it is enough to show that there exists $\gamma \in \Gamma_s^t$ for which $\left(\chi \downarrow_{G_\gamma}, 1_{G_\gamma} \right)_{G_\gamma} \neq 0$.

Several papers are devoted to calculating $\dim V_\chi^t(G)$ in a more closed form than (1). Cummings [1] in the case that G is a cyclic subgroup of S_t generated by a cycle of length t gives a formula for $\dim V_\chi^t(G)$. In [5] the case when G is a dihedral group of order $2t$ is considered and a formula is given when G is equal to the whole group S_t in [8] and [9]. Also in [10] a formula for calculating $\dim V_\chi^t(G)$ in the case that $G = \langle \pi_1 \rangle \cdots \langle \pi_p \rangle$ and in [2] for $G = \langle \pi_1 \cdots \pi_p \rangle$ is given, where π_i, $1 \leq i \leq p$, are disjoint cycles in S_t.

In this paper we compute $\dim V_\chi^t(G)$ for the projective special linear group of degree 2 over a field with q elements, namely we let $G = PSL_2(q)$, where q is a power of a prime number. We also discuss about when these vector spaces are nonzero.

2 Projective Special Linear Group

The special linear group of degree 2 is denoted by $SL_2(q)$, where $q = p^n$ and p is a prime number and n is a nonnegative integer. The character table of this group when $p = 2$ or p is an odd prime is given in [3]. We use these characters with the same name as in [3].

It is well-known that, if $N \lhd G$ and χ is a character of G with $N \subseteq \ker\chi$ and if $\hat{\chi}$ is a function defined by $\hat{\chi}(gN) = \chi(g)$, then $\hat{\chi}$ is a character of G/N. Conversely if $\hat{\chi}$ is a character of G/N, then the function χ defined by $\chi(g) = \hat{\chi}(gN)$ is a character of G having N in its kernel. In both cases $\chi \in I(G)$, $N \subseteq \ker\chi$, if and only if $\hat{\chi} \in I(G/N)$, (see [6]). Since the projective special linear group of degree 2, $G = PSL_2(q)$, where $q = p^n$, p is a prime number, n is a nonnegative integer, is a quotient of $SL_2(q)$, we can compute the character table of $G = PSL_2(q)$. These are given in Tables 1, 2 and 3.

It is well-known that $G = PSL_2(q)$ acts faithfully and 2-transitively on the $q+1$ points of the projective line Ω and so we can assume that $G = PSL_2(q)$ is a subgroup of S_{q+1}, therefore $V_\chi^{q+1}(G)$ is meaningful for any $\chi \in I(G)$ and we want to compute the dimensions of these vector spaces for all χ given in Tables 1, 2 and 3. By formula (1), to do this, we need to know the permutation structure of the elements of G.

Now we want to obtain the permutation structure of each element of $G = PSL_2(q)$ as a subgroup of S_{q+1}, but since elements in the same conjugacy class have the same permutation structure, we obtain the permutation structure of a representative from each conjugacy class of elements of G.

For $g \in G$, let $\mathrm{fix}(g) = \{i \mid 1 \le i \le q+1, g(i) = i\}$. Then θ defined by $\theta(g) = |\mathrm{fix}(g)|$, $g \in G$, is the permutation character of G acting on the set of points. Since G acts 2-transitively on the set of points, it is well-known that $\nu : G \to \mathbb{C}$ defined by $\nu(g) = |\mathrm{fix}(g)| - 1$, $g \in G$, is an irreducible character of G, (see [6]). In the case $G = PSL_2(q)$; q is odd, $q \overset{4}{\equiv} 1$; ν is equal to one of the irreducible characters in Table 1. Since $\nu(\{-I, I\}1) = (q+1) - 1 = q$, ν must be equal to ψ, so $|\mathrm{fix}(g)| = 1 + \psi(g)$, $g \in G$, from which we obtain the $|\mathrm{fix}(g)|$ row in Table 4. In the other cases we obtain the $|\mathrm{fix}(g)|$ row in Table 5 and 6. Therefore using $|\mathrm{fix}(g)|$ row and $o(g)$ row in Tables 4, 5 and 6, we obtain the permutation structure of elements of $G = PSL_2(q)$. Perhaps a few words may be necessary to explain the cycle structure of elements of $PSL_2(q)$. We use [3] for the matrix shapes of a, b, c and d as they appear in Tables 4, 5 and 6. In the case $G = PSL_2(q)$; q is odd, $q \overset{4}{\equiv} 1$; if an element x has order r and r is a prime number then all the non-trivial cycles appearing in the cycle structure of x must have length r from which the cycle structure of $\{-I, I\}c$, $\{-I, I\}d$ and $\{-I, I\}a^{(q-1)/4}$ that have orders p, p and 2 respectively follows. For $\{-I, I\}a^l$ we note that $\{-I, I\}a$ has two fixed points and two cycles of length $(q-1)/4$. Now according to the properties of permutations we can find

Table 1. The character table of $G = PSL_2(q)$; q is odd, $q = p^n$, $q \overset{4}{\equiv} 1$; $|G| = \frac{1}{2}q(q^2-1)$

g	$\{-I, I\}1$	$\{-I, I\}c$	$\{-I, I\}d$	$\{-I, I\}a^l$	$\{-I, I\}a^{(q-1)/4}$	$\{-I, I\}b^m$
$\|(g)\|$	1	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$	$q(q+1)$	$\frac{1}{2}q(q+1)$	$q(q-1)$
$o(g)$	1	p	p	$\frac{(q-1)/2}{(l,(q-1)/2)}$	2	$\frac{(q+1)/2}{(m,(q+1)/2)}$
1_G	1	1	1	1	1	1
ψ	q	0	0	1	1	-1
χ_i	$q+1$	1	1	$\rho^{il}+\rho^{-il}$	$\rho^{i(q-1)/4}+\rho^{-i(q-1)/4}$	0
θ_j	$q-1$	-1	-1	0	0	$-(\sigma^{jm}+\sigma^{-jm})$
ξ_1	$\frac{1}{2}(q+1)$	$\frac{1}{2}(1+\sqrt{q})$	$\frac{1}{2}(1-\sqrt{q})$	$(-1)^l$	$(-1)^{(q-1)/4}$	0
ξ_2	$\frac{1}{2}\ (q+1)$	$\frac{1}{2}(1-\sqrt{q})$	$\frac{1}{2}(1+\sqrt{q})$	$(-1)^l$	$(-1)^{(q-1)/4}$	0

$i = 2, 4, 6, \ldots, (q-5)/2,$ $l = 1, 2, \ldots, (q-5)/4,$ $\rho = e^{2\pi\sqrt{-1}/q-1}$
$j = 2, 4, 6, \ldots, (q-1)/2,$ $m = 1, 2, \ldots, (q-1)/4,$ $\sigma = e^{2\pi\sqrt{-1}/q+1}$
$|I(G)| = 4 + \frac{q-5}{4} + \frac{q-1}{4} = \frac{q+5}{2}$

Table 2. The character table of $G = PSL_2(q)$; q is odd, $q = p^n$, $q \overset{4}{\equiv} 3$; $|G| = \frac{1}{2}q(q^2-1)$

g	$\{-I, I\}1$	$\{-I, I\}c$	$\{-I, I\}d$	$\{-I, I\}a^l$	$\{-I, I\}b^m$	$\{-I, I\}b^{(q+1)/4}$
$\|(g)\|$	1	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$	$q(q+1)$	$q(q-1)$	$\frac{1}{2}q(q-1)$
$o(g)$	1	p	p	$\frac{(q-1)/2}{(l,(q-1)/2)}$	$\frac{(q+1)/2}{(m,(q+1)/2)}$	2
1_G	1	1	1	1	1	1
ψ	q	0	0	1	-1	-1
χ_i	$q+1$	1	1	$\rho^{il}+\rho^{-il}$	0	0
θ_j	$q-1$	-1	-1	0	$-(\sigma^{jm}+\sigma^{-jm})$	$-(\sigma^{j(q+1)/4}+\sigma^{-j(q+1)/4})$
η_1	$\frac{1}{2}(q-1)$	$-\frac{1}{2}(1-\sqrt{-q})$	$-\frac{1}{2}(1+\sqrt{-q})$	0	$(-1)^{m+1}$	$(-1)^{(q+5)/4}$
η_2	$\frac{1}{2}(q-1)$	$-\frac{1}{2}(1+\sqrt{-q})$	$-\frac{1}{2}(1-\sqrt{-q})$	0	$(-1)^{m+1}$	$(-1)^{(q+5)/4}$

$i = 2, 4, 6, \ldots, (q-3)/2,$ $l = 1, 2, 3, \ldots, (q-3)/4,$ $\rho = e^{2\pi\sqrt{-1}/q-1}$
$j = 2, 4, 6, \ldots, (q-3)/2,$ $m = 1, 2, 3, \ldots, (q-3)/4,$ $\sigma = e^{2\pi\sqrt{-1}/q+1}$
$|I(G)| = 4 + \frac{q-3}{4} + \frac{q-3}{4} = \frac{q+5}{2}$

Table 3. The character table of $G = PSL_2(q)$; q is even, $q = 2^n$; $|G| = q(q^2-1)$

g	$\{I\}1$	$\{I\}c$	$\{I\}a^l$	$\{I\}b^m$
$\|(g)\|$	1	q^2-1	$q(q+1)$	$q(q-1)$
$o(g)$	1	2	$\frac{q-1}{(l,q-1)}$	$\frac{q+1}{(m,q+1)}$
1_G	1	1	1	1
ψ	q	0	1	-1
χ_i	$q+1$	1	$\rho^{il}+\rho^{-il}$	0
θ_j	$q-1$	-1	0	$-(\sigma^{jm}+\sigma^{-jm})$

$1 \le i \le (q-2)/2$, $\quad 1 \le l \le (q-2)/2$, $\quad \rho = e^{2\pi\sqrt{-1}/q-1}$
$1 \le j \le q/2$, $\quad 1 \le m \le q/2$, $\quad \sigma = e^{2\pi\sqrt{-1}/q+1}$
$|I(G)| = 2 + \frac{q-2}{2} + \frac{q}{2} = q+1$

the cycle structures of powers of $\{-I, I\}a$. The element $\{-I, I\}b$ is a Singer-cycle and consists of only one cycle and the cycle structures of its powers is clear. The discussion is similar in the other cases.

3 On the Dimensions of Symmetry Classes of Tensors Associated with $G = PSL_2(q)$

As we remarked earlier the group $G = PSL_2(q)$ acts faithfully on the set of the one-dimensional subspaces $\langle v\rangle$ of $\mathbb{V}_2(q)$. Therefore we regard G as a subgroup of the symmetric group on the $q+1$ letters and find the dimensions of the symmetry classes of tensors associated with each irreducible character of G. Note that the names of the irreducible characters of $G = PSL_2(q)$ are as indicated in [3]. In the following (a, b) denotes the greatest common divisor of a and b; and ρ and σ are primitive $(q-1)$th and $(q+1)$th roots of unity in $\mathbb{C}$ respectively.

Theorem 1 *Let $G = PSL_2(q)$ as a subgroup of S_{q+1}; where q is odd, $q = p^n$, $q \overset{4}{\equiv} 1$; and let V be an s-dimensional vector space over the complex field $\mathbb{C}$. Then we have the following formulae for the dimension of $V_\chi^{q+1}(G)$ where $\chi \in I(G)$.*

$$\dim V_{1_G}^{q+1}(G) = \frac{2}{q(q^2-1)}\Bigg[s^{q+1} + (q^2-1)s^{1+p^{n-1}} + q(q+1)\sum_{l=1}^{(q-5)/4} s^{2+2(l,(q-1)/2)} + \frac{1}{2}q(q+1)s^{(q+3)/2} + q(q-1)\sum_{m=1}^{(q-1)/4} s^{2(m,(q+1)/2)}\Bigg] ,$$

Table 4. The permutation structure of elements of the group $G = PSL_2(q) \leq S_{q+1}$; q is odd, $q = p^n$, $q \overset{4}{\equiv} 1$; $|G| = \frac{1}{2}q(q^2-1)$

g	$\{-I, I\}1$	$\{-I, I\}c$	$\{-I, I\}d$	$\{-I, I\}a^l$	$\{-I, I\}a^{(q-1)/4}$	$\{-I, I\}b^m$
$\|(g)\|$	1	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$	$q(q+1)$	$\frac{1}{2}q(q+1)$	$q(q-1)$
$o(g)$	1	p	p	$\frac{(q-1)/2}{(l,(q-1)/2)}$	2	$\frac{(q+1)/2}{(m,(q+1)/2)}$
$\|\mathrm{fix}(g)\|$	$q+1$	1	1	2	2	0
per. stru. of g	1^{q+1}	$1^1 p^{p^{n-1}}$	$1^1 p^{p^{n-1}}$	$1^2 {\frac{(q-1)/2}{(l,(q-1)/2)}}^{2(l,(q-1)/2)}$	$1^2 2^{(q-1)/2}$	${\frac{(q+1)/2}{(m,(q+1)/2)}}^{2(m,(q+1)/2)}$

$1 \leq l \leq (q-5)/4$
$1 \leq m \leq (q-1)/4$

Table 5. The permutation structure of elements of the group $G = PSL_2(q) \leq S_{q+1}$; q is odd, $q = p^n$, $q \overset{4}{\equiv} 3$; $|G| = \frac{1}{2}q(q^2-1)$

g	$\{-I, I\}1$	$\{-I, I\}c$	$\{-I, I\}d$	$\{-I, I\}a^l$	$\{-I, I\}b^m$	$\{-I, I\}b^{(q+1)/4}$
$\|(g)\|$	1	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$	$q(q+1)$	$q(q-1)$	$\frac{1}{2}q(q-1)$
$o(g)$	1	p	p	$\frac{(q-1)/2}{(l,(q-1)/2)}$	$\frac{(q+1)/2}{(m,(q+1)/2)}$	2
$\|\mathrm{fix}(g)\|$	$q+1$	1	1	2	0	0
per. stru. of g	1^{q+1}	$1^1 p^{p^{n-1}}$	$1^1 p^{p^{n-1}}$	$1^2 {\frac{(q-1)/2}{(l,(q-1)/2)}}^{2(l,(q-1)/2)}$	${\frac{(q+1)/2}{(m,(q+1)/2)}}^{2(m,(q+1)/2)}$	$2^{(q+1)/2}$

$1 \leq l \leq (q-3)/4$
$1 \leq m \leq (q-3)/4$

Table 6. The permutation structure of elements of the group $G = PSL_2(q) \leq S_{q+1}$; q is even, $q = 2^n$; $|G| = q(q^2-1)$

g	$\{I\}1$	$\{I\}c$	$\{I\}a^l$	$\{I\}b^m$
$\lvert(g)\rvert$	1	q^2-1	$q(q+1)$	$q(q-1)$
$o(g)$	1	2	$\frac{q-1}{(l,q-1)}$	$\frac{q+1}{(m,q+1)}$
$\lvert\text{fix}(g)\rvert$	$q+1$	1	2	0
per. stru. of g	1^{q+1}	$1^1 2^{q/2}$	$1^2 \, \frac{q-1}{(l,q-1)}^{(l,q-1)}$	$\frac{q+1}{(m,q+1)}^{(m,q+1)}$

$1 \leq l \leq (q-2)/2$
$1 \leq m \leq q/2$

$$\dim V_{\psi}^{q+1}(G) = \frac{2}{q^2-1}\Bigg[qs^{q+1} + q(q+1)\sum_{l=1}^{(q-5)/4} s^{2+2(l,(q-1)/2)} + \frac{1}{2}q(q+1)s^{(q+3)/2} - q(q-1)\sum_{m=1}^{(q-1)/4} s^{2(m,(q+1)/2)}\Bigg] ,$$

$$\dim V_{\chi_i}^{q+1}(G) = \frac{2}{q(q-1)}\Bigg[(q+1)s^{q+1} + (q^2-1)s^{1+p^{n-1}} + q(q+1)\sum_{l=1}^{(q-5)/4} (\rho^{il} + \rho^{-il})s^{2+2(l,(q-1)/2)} + \frac{1}{2}q(q+1)(\rho^{i(q-1)/4} + \rho^{-i(q-1)/4})s^{(q+3)/2}\Bigg] ,$$

$$i = 2, 4, 6, \ldots, (q-5)/2$$

$$\dim V_{\theta_j}^{q+1}(G) = \frac{2}{q(q+1)}\Bigg[(q-1)s^{q+1} - (q^2-1)s^{1+p^{n-1}} - q(q-1)\sum_{m=1}^{(q-1)/4} (\sigma^{jm} + \sigma^{-jm})s^{2(m,(q+1)/2)}\Bigg] ,$$

$$j = 2, 4, 6, \ldots, (q-1)/2$$

$$\dim V_{\xi_1}^{q+1}(G) = \frac{1}{q(q-1)}\Bigg[\frac{1}{2}(q+1)s^{q+1} + \frac{1}{2}(q^2-1)s^{1+p^{n-1}} + q(q+1)\sum_{l=1}^{(q-5)/4} (-1)^l s^{2+2(l,(q-1)/2)} + (-1)^{(q-1)/4}\frac{1}{2}q(q+1)s^{(q+3)/2}\Bigg] ,$$

$$\begin{aligned}\dim V_{\xi_2}^{q+1}(G) = \frac{1}{q(q-1)}\Bigg[&\frac{1}{2}(q+1)s^{q+1} + \frac{1}{2}(q^2-1)s^{1+p^{n-1}}\\ &+ q(q+1)\sum_{l=1}^{(q-5)/4}(-1)^l s^{2+2(l,(q-1)/2)}\\ &+ (-1)^{(q-1)/4}\frac{1}{2}q(q+1)s^{(q+3)/2}\Bigg].\end{aligned}$$

Proof. It is clear that if π is a cycle of length a and $(a,k)=d$, then π^k has d cycles of length a/d, therefore $c(\pi^k)=d=(a,k)$. Now the permutation structure of the elements of $G=PSL_2(q)$ when acting on the $q+1$ one-dimensional subspaces are found in Table 4 and from formula (1) the theorem follows. Note that in any case $PSL_2(q)$ acts 2-transitively on the $q+1$ projective points and its permutation character θ is given by $\theta=1+\psi$. In fact $\theta(g)=1+\psi(g)$ is equal to the number of points left fixed by $g\in PSL_2(q)$ which can be computed from Table 1 and which is indicated by fix(g). □

Similar to the above, we obtain Theorem 2 and 3 below.

Theorem 2 *Let $G=PSL_2(q)$ as a subgroup of S_{q+1}; where q is odd, $q=p^n$, $q\overset{4}{\equiv}3$; and let V be an s-dimensional vector space over the complex field $\mathbb{C}$. Then we have the following formulae for the dimension of $V_\chi^{q+1}(G)$, where $\chi\in I(G)$.*

$$\begin{aligned}\dim V_{1_G}^{q+1}(G) = \frac{2}{q(q^2-1)}\Bigg[&s^{q+1} + (q^2-1)s^{1+p^{n-1}}\\ &+ q(q+1)\sum_{l=1}^{(q-3)/4}s^{2+2(l,(q-1)/2)}\\ &+ q(q-1)\sum_{m=1}^{(q-3)/4}s^{2(m,(q+1)/2)} + \frac{1}{2}q(q-1)s^{(q+1)/2}\Bigg],\end{aligned}$$

$$\begin{aligned}\dim V_{\psi}^{q+1}(G) = \frac{2}{q^2-1}\Bigg[&qs^{q+1} + q(q+1)\sum_{l=1}^{(q-3)/4}s^{2+2(l,(q-1)/2)}\\ &- q(q-1)\sum_{m=1}^{(q-3)/4}s^{2(m,(q+1)/2)} - \frac{1}{2}q(q-1)s^{(q+1)/2}\Bigg],\end{aligned}$$

$$\begin{aligned}\dim V_{\chi_i}^{q+1}(G) = \frac{2}{q(q-1)}\Bigg[&(q+1)s^{q+1} + (q^2-1)s^{1+p^{n-1}}\\ &+ q(q+1)\sum_{l=1}^{(q-3)/4}(\rho^{il}+\rho^{-il})s^{2+2(l,(q-1)/2)}\Bigg],\\ &i=2,4,6,\ldots,(q-3)/2\end{aligned}$$

$$\begin{aligned}\dim V_{\theta_j}^{q+1}(G) = \frac{2}{q(q+1)}\Bigg[&(q-1)s^{q+1} - (q^2-1)s^{1+p^{n-1}} \\ &- q(q-1)\sum_{m=1}^{(q-3)/4}(\sigma^{jm}+\sigma^{-jm})s^{2(m,(q+1)/2)} \\ &- \frac{1}{2}q(q-1)(\sigma^{j(q+1)/4}+\sigma^{-j(q+1)/4})s^{(q+1)/2}\Bigg], \\ &\qquad j = 2, 4, 6, \ldots, (q-3)/2\end{aligned}$$

$$\begin{aligned}\dim V_{\eta_1}^{q+1}(G) = \frac{1}{q(q+1)}\Bigg[&\frac{1}{2}(q-1)s^{q+1} - \frac{1}{2}(q^2-1)s^{1+p^{n-1}} \\ &- q(q-1)\sum_{m=1}^{(q-3)/4}(-1)^m s^{2(m,(q+1)/2)} \\ &+ (-1)^{(q+5)/4}\frac{1}{2}q(q-1)s^{(q+1)/2}\Bigg],\end{aligned}$$

$$\begin{aligned}\dim V_{\eta_2}^{q+1}(G) = \frac{1}{q(q+1)}\Bigg[&\frac{1}{2}(q-1)s^{q+1} - \frac{1}{2}(q^2-1)s^{1+p^{n-1}} \\ &- q(q-1)\sum_{m=1}^{(q-3)/4}(-1)^m s^{2(m,(q+1)/2)} \\ &+ (-1)^{(q+5)/4}\frac{1}{2}q(q-1)s^{(q+1)/2}\Bigg].\end{aligned}$$

Theorem 3 *Let $G = PSL_2(q)$ as a subgroup of S_{q+1}; where q is even, $q = 2^n$; and let V be an s-dimensional vector space over the complex field $\mathbb{C}$. Then we have the following formulae for the dimension of $V_\chi^{q+1}(G)$, where $\chi \in I(G)$.*

$$\begin{aligned}\dim V_{1_G}^{q+1}(G) = \frac{1}{q(q^2-1)}\Bigg[&s^{q+1} + (q^2-1)s^{(q+2)/2} \\ &+ q(q+1)\sum_{l=1}^{(q-2)/2} s^{2+(l,q-1)} \\ &+ q(q-1)\sum_{m=1}^{q/2} s^{(m,q+1)}\Bigg],\end{aligned}$$

$$\begin{aligned}\dim V_{\psi}^{q+1}(G) = \frac{1}{q^2-1}\Bigg[&qs^{q+1} + q(q+1)\sum_{l=1}^{(q-2)/2} s^{2+(l,q-1)} \\ &- q(q-1)\sum_{m=1}^{q/2} s^{(m,q+1)}\Bigg],\end{aligned}$$

$$\begin{aligned}\dim V_{\chi_i}^{q+1}(G) = \frac{1}{q(q-1)}\Bigg[&(q+1)s^{q+1} + (q^2-1)s^{(q+2)/2} \\ &+ q(q+1)\sum_{l=1}^{(q-2)/2}(\rho^{il}+\rho^{-il})s^{2+(l,q-1)}\Bigg], \\ &\qquad\qquad i = 1, 2, \ldots, (q-2)/2\end{aligned}$$

$$\begin{aligned}\dim V_{\theta_j}^{q+1}(G) = \frac{1}{q(q+1)}\Bigg[&(q-1)s^{q+1} - (q^2-1)s^{(q+2)/2} \\ &- q(q-1)\sum_{m=1}^{q/2}(\sigma^{jm}+\sigma^{-jm})s^{(m,q+1)}\Bigg], \\ &\qquad\qquad j = 1, 2, \ldots, q/2\end{aligned}$$

4 On the Vanishing of Symmetry Classes of Tensors Associated with $G = PSL_2(q)$

In this section, we discuss the question of when the symmetry classes of tensors associated with $G = PSL_2(q)$ are nonzero vector spaces. If $\dim V = s = 1$, then it is clear that for all χ, $\chi \in I(G) - \{1_G\}$, $V_\chi^{q+1}(G) = 0$ and $V_{1_G}^{q+1}(G) \neq 0$. Therefore we deal with the case $\dim V = s = 2$ in the following theorem.

Theorem 4 *Consider $G = PSL_2(q)$ as a subgroup of S_{q+1} and let V be a vector space over the complex field $\mathbb{C}$, such that* $\dim V = s = 2$.

(*a*) *If q is odd, $q = p^n$, $q \overset{4}{\equiv} 1$; then for all χ, $\chi \in I(G) - \{\chi_i, \xi_1, \xi_2 \mid i = 2, 4, \ldots, (q-5)/2; i \overset{4}{\equiv} 2\}$, we have $V_\chi^{q+1}(G) \neq 0$. Additionally if $q \overset{8}{\equiv} 1$, then $V_{\xi_1}^{q+1}(G) \neq 0$ and $V_{\xi_2}^{q+1}(G) \neq 0$,*

(*b*) *If q is odd, $q = p^n$, $q \overset{4}{\equiv} 3$; then for all χ, $\chi \in I(G) - \{\theta_j, \eta_1, \eta_2 \mid j = 2, 4, \ldots, (q-3)/2; j \overset{4}{\equiv} 0\}$, we have $V_\chi^{q+1}(G) \neq 0$. Additionally if $q \overset{8}{\equiv} 3$, then $V_{\eta_1}^{q+1}(G) \neq 0$ and $V_{\eta_2}^{q+1}(G) \neq 0$,*

(*c*) *If q is even, $q = 2^n$; then for all χ, $\chi \in I(G) - \{\theta_j \mid 1 \leq j \leq q/2\}$, we have $V_\chi^{q+1}(G) \neq 0$.*

Proof. Let $\gamma = (1, 1, 2, 2, \ldots, 2) \in \Gamma_2^{q+1}$ and consider the action of G on the set of 2-subsets of Ω consisting of the $q+1$ points; $\Omega^{\{2\}}$. This action is transitive and let $G_{\{\tilde{\Omega}\}}$, $\tilde{\Omega} \subseteq \Omega$, $|\tilde{\Omega}| = 2$, denote the setwise stabilizer of $\tilde{\Omega}$. It is easy to see that $G_\gamma = G_{\{\tilde{\Omega}\}}$. By Frobenius reciprocity we have $(\chi \downarrow_{G_\gamma}, 1_{G_\gamma})_{G_\gamma} = (\chi, 1_{G_\gamma} \uparrow^G)_G$. But $1_{G_\gamma} \uparrow^G = 1_{G_{\{\tilde{\Omega}\}}} \uparrow^G = \xi$ is the permutation character of G acting on $\Omega^{\{2\}}$. So

$$(\chi \downarrow_{G_\gamma}, 1_{G_\gamma})_{G_\gamma} = (\chi, \xi)_G \tag{5}$$

where $\xi(g)$ is the number of 2-subsets of Ω fixed by g. Consider g as a permutation on Ω and recall that the permutation character of G on Ω is denoted by θ. Therefore there are $\binom{\theta(g)}{2}$ subsets of Ω of size 2 which are fixed by g setwise. A simple calculation shows that there are $\frac{1}{2}(\theta(g^2)-\theta(g))$ cycles of length 2 in the cycle structure of g and therefore the total number of 2-subsets of Ω fixed by g is

$$\xi(g) = \binom{\theta(g)}{2} + \frac{\theta(g^2)-\theta(g)}{2} = \frac{1}{2}\left(\theta(g)^2+\theta(g^2)\right) - \theta(g) \ .$$

Using Tables 4, 5 and 6 we computed the values of ξ which are given in Tables 7, 8 and 9.

If $q \overset{4}{\equiv} 1$, then the following decomposition of ξ is computed using Tables 1 and 7,

$$\xi = \begin{cases} 1_G + 2\psi + 2\sum\limits_{i \overset{4}{\equiv} 0} \chi_i + \sum\limits_{i \text{ odd}} \chi_i + \sum\limits_{j} \theta_j + \xi_1 + \xi_2 & \text{if } q \overset{8}{\equiv} 1, \\ 1_G + 2\psi + 2\sum\limits_{i \overset{4}{\equiv} 0} \chi_i + \sum\limits_{i \text{ odd}} \chi_i + \sum\limits_{j} \theta_j & \text{otherwise} \ . \end{cases}$$

Therefore by (5) if $\chi \in I(G) - \{\chi_i, \xi_1, \xi_2 \mid i = 2, 4, \ldots, (q-5)/2;\ i \overset{4}{\equiv} 2\}$, then $(\chi \downarrow_{G_\gamma}, 1_{G_\gamma})_{G_\gamma} \neq 0$ and so $V_\chi^{q+1}(G) \neq 0$, additionally if $q \overset{8}{\equiv} 1$, then

Table 7. $G = PSL_2(q)$; q is odd, $q = p^n$, $q \overset{4}{\equiv} 1$

g	$\{-I, I\}1$	$\{-I, I\}c$	$\{-I, I\}d$	$\{-I, I\}a^l$	$\{-I, I\}a^{(q-1)/4}$	$\{-I, I\}b^m$
$\lvert C_G(g)\rvert$	$\frac{1}{2}q(q^2-1)$	q	q	$\frac{1}{2}(q-1)$	$q-1$	$\frac{1}{2}(q+1)$
$\theta(g)$	$q+1$	1	1	2	2	0
$\xi(g)$	$\frac{1}{2}q(q+1)$	0	0	1	$\frac{1}{2}(q+1)$	0
$\xi'(g)$	$q(q+1)$	0	0	2	2	0

$l = 1, 2, \ldots, (q-5)/4$
$m = 1, 2, \ldots, (q-1)/4$

Table 8. $G = PSL_2(q)$; q is odd, $q = p^n$, $q \overset{4}{\equiv} 3$

g	$\{-I, I\}\,1$	$\{-I, I\}c$	$\{-I, I\}d$	$\{-I, I\}a^l$	$\{-I, I\}b^m$	$\{-I, I\}b^{(q+1)/4}$
$\lvert C_G(g)\rvert$	$\frac{1}{2}q(q^2-1)$	q	q	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q+1)$	$q+1$
$\theta(g)$	$q+1$	1	1	2	0	0
$\xi(g)$	$\frac{1}{2}q(q+1)$	0	0	1	0	$\frac{1}{2}(q+1)$
$\xi'(g)$	$q(q+1)$	0	0	2	0	0

$l = 1, 2, \ldots, (q-3)/4$
$m = 1, 2, \ldots, (q-3)/4$

Table 9. $G = PSL_2(q)$; q is even, $q = 2^n$

g	$\{I\}1$	$\{I\}c$	$\{I\}a^l$	$\{I\}b^m$
$\lvert C_G(g)\rvert$	$q(q^2-1)$	q	$q-1$	$q+1$
$\theta(g)$	$q+1$	1	2	0
$\xi(g)$	$\frac{1}{2}q(q+1)$	$\frac{1}{2}q$	1	0
$\xi'(g)$	$q(q+1)$	0	2	0

$1 \le l \le (q-2)/2$
$1 \le m \le q/2$

$(\xi_1 \downarrow_{G_\gamma}, 1_{G_\gamma})_{G_\gamma} \neq 0$ and $(\xi_2 \downarrow_{G_\gamma}, 1_{G_\gamma})_{G_\gamma} \neq 0$ and so $V_{\xi_1}^{q+1}(G) \neq 0$ and $V_{\xi_2}^{q+1}(G) \neq 0$.

Furthermore, using Tables 2 and 8 when $q \overset{4}{\equiv} 3$ we are able to decompose ξ as follows,

$$\xi = \begin{cases} 1_G + \psi + \sum_i \chi_i + 2\sum_{j \overset{4}{\equiv} 2} \theta_j + \sum_{j \text{ odd}} \theta_j + \eta_1 + \eta_2 & \text{if } q \overset{8}{\equiv} 3, \\ 1_G + \psi + \sum_i \chi_i + 2\sum_{j \overset{4}{\equiv} 2} \theta_j + \sum_{j \text{ odd}} \theta_j & \text{otherwise} \end{cases}.$$

Therefore by (5) if $\chi \in I(G) - \{\theta_j, \eta_1, \eta_2 \mid j = 2, 4, \ldots, (q-3)/2;\ j \overset{4}{\equiv} 0\}$, then $(\chi \downarrow_{G_\gamma}, 1_{G_\gamma})_{G_\gamma} \neq 0$ and so $V_\chi^{q+1}(G) \neq 0$, additionally if $q \overset{8}{\equiv} 3$, then $(\eta_1 \downarrow_{G_\gamma}, 1_{G_\gamma})_{G_\gamma} \neq 0$ and $(\eta_2 \downarrow_{G_\gamma}, 1_{G_\gamma})_{G_\gamma} \neq 0$ and so $V_{\eta_1}^{q+1}(G) \neq 0$ and $V_{\eta_2}^{q+1}(G) \neq 0$.

Also if q is even, then by Tables 3 and 9 we have

$$\xi = 1_G + \psi + \sum_i \chi_i.$$

Therefore by (5) if $\chi \in I(G) - \{\theta_j \mid 1 \le j \le q/2\}$, then $(\chi \downarrow_{G_\gamma}, 1_{G_\gamma})_{G_\gamma} \neq 0$ and so $V_\chi^{q+1}(G) \neq 0$. □

Theorem 5 *Consider* $G = PSL_2(q)$ *as a subgroup of* S_{q+1} *and let* V *be a vector space over the complex field* $\mathbb{C}$, *such that* $\dim V = s \ge 3$. *Then for all* χ, $\chi \in I(G)$, *we have* $V_\chi^{q+1}(G) \neq 0$.

Proof. First we assume that $\dim V = s = 3$. Let $\gamma = (1, 2, 3, 3, \ldots, 3) \in \Gamma_3^{q+1}$ and consider the action of G on the set of ordered pairs of points of Ω namely $\Omega^{(2)}$. This action is transitive and let $G_{(\hat{\Omega})}$, $\hat{\Omega} \subseteq \Omega$, $|\hat{\Omega}| = 2$, denote the pointwise stabilizer of $\hat{\Omega}$. Therefore $G_\gamma = G_{(\hat{\Omega})}$. By Frobenius reciprocity we have $(\chi \downarrow_{G_\gamma}, 1_{G_\gamma})_{G_\gamma} = (\chi, 1_{G_\gamma} \uparrow^G)_G$. But $1_{G_\gamma} \uparrow^G = 1_{G_{(\hat{\Omega})}} \uparrow^G = \xi'$ is the permutation character of G acting on $\Omega^{(2)}$. So

$$\left(\chi\downarrow_{G_\gamma}, 1_{G_\gamma}\right)_{G_\gamma} = (\chi, \xi')_G \tag{6}$$

where $\xi'(g)$ is the number of ordered pairs fixed by g and so using similar techniques as in the previous theorem we get

$$\xi'(g) = 2\binom{\theta(g)}{2} = \theta(g)^2 - \theta(g)\ .$$

We computed the values of ξ' in Tables 7, 8 and 9. Using these tables and the character table of $G = PSL_2(q)$ we obtain

$$\xi' = 1_G + 3\psi + 2\sum_i \chi_i + 2\sum_j \theta_j + \xi_1 + \xi_2\ ,$$

when $q \overset{4}{\equiv} 1$ and therefore by (6) if $\chi \in I(G)$, then $(\chi\downarrow_{G_\gamma}, 1_{G_\gamma})_{G_\gamma} \neq 0$ and so $V_\chi^{q+1}(G) \neq 0$. If $q \overset{4}{\equiv} 3$, then

$$\xi' = 1_G + 3\psi + 2\sum_i \chi_i + 2\sum_j \theta_j + \eta_1 + \eta_2\ .$$

Therefore by (6) if $\chi \in I(G)$, then $(\chi\downarrow_{G_\gamma}, 1_{G_\gamma})_{G_\gamma} \neq 0$ and so $V_\chi^{q+1}(G) \neq 0$. If q is even, then

$$\xi' = 1_G + 2\psi + \sum_i \chi_i + \sum_j \theta_j\ .$$

Therefore by (6) if $\chi \in I(G)$, then $(\chi\downarrow_{G_\gamma}, 1_{G_\gamma})_{G_\gamma} \neq 0$ and so $V_\chi^{q+1}(G) \neq 0$. So we proved that if $\dim V = s = 3$, then $V_\chi^{q+1}(G) \neq 0$, for all χ, $\chi \in I(G)$.

Now we assume that $\dim V = s \geq 4$. In this case if we consider $\gamma = (1, 2, 3, 4, 4, \ldots, 4) \in \Gamma_s^{q+1}$, then by Tables 4, 5 and 6 the elements of G; except the identity, fix at most 2 points of Ω and so G_γ is the trivial subgroup of G, i.e., $G_\gamma = \{1\}$; therefore for all χ, $\chi \in I(G)$, $(\chi\downarrow_{G_\gamma}, 1_{G_\gamma})_{G_\gamma} = \chi(1) \neq 0$ and so $V_\chi^{q+1}(G) \neq 0$. □

References

1. L. J. Cummings: Cyclic Symmetry Classes, J. Algebra **40**, 401–405 (1976)
2. M. R. Darafsheh, M. R. Pournaki: On the Dimensions of Cyclic Symmetry Classes of Tensors, J. Algebra **205**, 317–325 (1998)
3. L. Dornhoff: Group Representation Theory, Part I, New York: Marcel Dekker, Inc., 1972
4. R. Freese: Inequalities for Generalized Matrix Functions Based on Arbitrary Characters, Linear Algebra Appl. **7**, 337–345 (1973)
5. R. R. Holmes, T. Y. Tam: Symmetry Classes of Tensors Associated with Certain Groups, Linear and Multilinear Algebra **32**, 21–31 (1992)
6. I. M. Isaacs: Character Theory of Finite Groups, Dover: Academic Press 1994

7. M. Marcus: Finite Dimensional Multilinear Algebra, Part 1, New York, Marcel Dekker, 1973
8. R. Merris: The Dimension of Certain Symmetry Classes of Tensors II, Linear and Multilinear Algebra **4**, 205–207 (1976)
9. R. Merris, M. A. Rashid: The Dimension of Certain Symmetry Classes of Tensors, Linear and Multilinear Algebra **2**, 245–248 (1974)
10. T. Y. Tam: On the Cyclic Symmetry Classes, J. Algebra **182**, 557–560 (1996)